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Solution of the Diophantine Equation of the form $a^x - b^y c^z = \pm 1$

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Abstract: *Our main goal in this paper is to find the new results of some exponential Diophantine equations of the form* $a^x - b^y c^z = \pm 1$. *We use the method of congruence with single modulus to find the new results for some Diophantine equations of this type.*

Introduction

In the theory of finite nonabelian simple groups, there are many applications in Alex [2] and [3] of the exponential Diophantine equations of the following form

$$a^x - b^y c^z = \pm 1 \qquad (1)$$

where a, b, c are positive integers and unknowns x, y, z are nonnegative integers.

In Alex [3], author uses the congruence with a single modulus to obtain all the solutions for the following exponential Diophantine equations:

$$2^x - 3^y 7^z = \pm 1 \qquad (2)$$

$$3^x - 2^y 7^z = \pm 1 \qquad (3)$$

and

$$7^x - 2^y 3^z = \pm 1 \qquad (4)$$

In Alex [2], author uses the same method to solve the following Diophantine equations:

$$2^x - 3^y 5^z = \pm 1 \qquad (5)$$

$$2^x - 5^y 7^z = \pm 1 \qquad (6)$$

$$3^x - 2^y 5^z = \pm 1 \qquad (7)$$

$$5^x - 2^y 3^z = \pm 1 \qquad (8)$$

$$5^x - 2^y 7^z = \pm 1 \qquad (9)$$

and

$$7^x - 2^y 5^z = \pm 1 \qquad (10)$$

where the equations (7) & (8) are also solved in Brenner [4].

Acu [1] solves the following Diophantine equations of the form (1) by using the elementary method,

$$2^x - 3^y 11^z = \pm 1 \qquad (11)$$

$$2^x - 3^y 13^z = \pm 1 \qquad (12)$$

$$2^x - 3^y 17^z = \pm 1 \qquad (13)$$

$$2^x - 3^y 19^z = \pm 1 \qquad (14)$$

Teng [5] uses the same method to prove that the equation

$$a^x - p_1^{y_1} p_2^{y_2} p_3^{y_3} \Lambda\Lambda\ p_k^{y_k} = \pm 1,$$

where a is a positive integer with $a > 1$ and $p_1, p_2, p_3, \Lambda\Lambda\ p_k$ are distinct primes with $g.c.d\ (a, p_1, p_2, p_3, \Lambda\Lambda\ p_k) = 1$, has only finitely many positive integer solutions $(x, y_1, y_2, y_3, \Lambda\Lambda\ y_k)$.

We will use the same method of congruence with a single modulus to solve some new Diophantine equations of the form (1).

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New Results

Theorem 1. *For the prime numbers* $c \in \{23, 29, 31, 37, 41\}$ *the Diophantine equation*

$$2^x - 3^y c^z = 1$$

has nonnegative integral common solutions: $(x, y, z) = (1,0,0)$ *and* $(2,1,0)$ *but when* $c = 31$, $(5,0,1)$ *is also a solution together with the common solutions.*

Proof: The theorem is proved in the following cases.

Case I: $c = 23$

For $z \geq 1$, we find by using *mod* 23 that $2^{11} \equiv 1 \pmod{23}$, which implies $x \equiv 0 \pmod{11}$. But, if we use *mod* 89, we also get $2^{11} \equiv 1 \pmod{89}$, which is a contradiction and so there is no solution.

Suppose $y \geq 3$, we find by using *mod* 27 that $2^{18} \equiv 1 \pmod{27}$ and this implies $x \equiv 0 \pmod{18}$. But, if we use *mod* 19, we also get $2^{18} \equiv 1 \pmod{19}$, which is a contradiction and so there is no solution.

Case II: $c = 29$

For $z \geq 1$, using *mod* 29 we find $x \equiv 0 \pmod{28}$. But, if we use *mod* 5, we obtain a contradiction.

Considering $y \geq 3$, using *mod* 27, we obtain $x \equiv 0 \pmod{18}$. But, if we use *mod* 7, we find a contradiction and so there is no solution.

Case III: $c = 31$

For $z \geq 2$, using *mod* 961, we find $x \equiv 0 \pmod{155}$. Now if we use *mod* 311, we obtain a contradiction.

Considering $y \geq 3$, using *mod* 27, we obtain $x \equiv 0 \pmod{18}$. In this case we have a contradiction for *mod* 7 and so there is no solution.

Therefore,

If $z = 1$ and $y \leq 2$, we find the solution $(x, y, z) = (5,0,1)$.

Case IV: $c = 37$

For $z \geq 1$, using *mod* 37 we find $x \equiv 0 \pmod{36}$. Now, if we use *mod* 5, we find a contradiction.

Suppose $y \geq 3$, using *mod* 27 we obtain $x \equiv 0 \pmod{18}$. This yields a contradiction *mod* 7 and so there is no solution.

Case V: $c = 41$

For $z \geq 1$, using *mod* 41, we find $x \equiv 0 \pmod{20}$. Now if we use *mod* 5, we obtain a contradiction.

Considering $y \geq 3$, using *mod* 27, we obtain $x \equiv 0 \pmod{18}$. But, if we use *mod* 7, we find a contradiction and so there is no solution.

Therefore, we conclude from the above cases that,

If $z = 0$ and $y \leq 2$, we find the solutions $(x, y, z) = (1,0,0)$ and $(2,1,0)$,

and there is no other solution for $c \in \{23, 29, 37, 41\}$ when $z = 1$ and $y \leq 2$.

Theorem 2. *For the prime numbers* $c \in \{23, 29, 31, 37, 41\}$ *the Diophantine equation*

$$2^x - 3^y c^z = -1$$

has nonnegative integral solutions: $(x, y, z) = (1,1,0)$ *and* $(3,2,0)$.

Proof: We prove the theorem in following cases.

Case I: $c = 23$
Considering $z \geq 1$, we find by using *mod* 23 that

$$2^x \equiv 2,4,8,16,9,18,13,3,6,12,1 \pmod{23}$$

and $$3^y 23^z \equiv 0 \pmod{23}$$

but $$-1 \equiv -1 \pmod{23} \equiv 22 \pmod{23}$$

This is a contradiction and yields no solution for $z \geq 1$.
For $y \geq 3$, using *mod* 27, we obtain $x \equiv 0 \pmod 9$. But, if we use *mod* 19, we find a contradiction and so there is no solution.

Case II: $c = 29$
For $z \geq 1$, using *mod* 29, we find $x \equiv 0 \pmod{14}$. Now, if we use *mod* 5, we obtain a contradiction.
Suppose $y \geq 3$, using *mod* 27, we obtain $x \equiv 0 \pmod 9$. But, if we use *mod* 19, we find a contradiction and so there is no solution.

Case III: $c = 31$
Considering $z \geq 1$, we find by using *mod* 31 that

$$2^x \equiv 2,4,8,16,1 \pmod{31}$$

and $$3^y 31^z \equiv 0 \pmod{31}$$

but $$-1 \equiv -1 \pmod{31} \equiv 30 \pmod{31},$$

This yields a contradiction and so there is no solution for $z \geq 1$.
For $y \geq 3$, using *mod* 27, we obtain $x \equiv 0 \pmod 9$. But, if we use *mod* 19, we find a contradiction and so there is no solution.

Case IV: $c = 37$
Suppose $z \geq 1$, using *mod* 37, we find $x \equiv 0 \pmod{18}$. Now if we use *mod* 5, we find a contradiction.
For $y \geq 3$, using *mod* 27, we obtain $x \equiv 0 \pmod 9$. But, if we use *mod* 19, we find a contradiction and so there is no solution.

Case V: $c = 41$
For $z \geq 1$, using *mod* 41, we find $x \equiv 0 \pmod{10}$. Now if we use *mod* 5, we obtain a contradiction.
Assume $y \geq 3$, using *mod* 27, we obtain $x \equiv 0 \pmod 9$. But, if we use *mod* 19, we find a contradiction and so there is no solution.

Therefore, we conclude from the above cases that,
If $z = 0$ and $y \leq 2$, we find the solutions $(x, y, z) = (1,1,0)$ and $(3,2,0)$.
If $z = 1$ and $y \leq 2$, there is no other solution.

Theorem 3. *For the prime numbers* $c \in \{11,13,17,19,23\}$ *the Diophantine equation*

$$3^x - 2^y c^z = 1$$

has nonnegative integral common solutions: $(x, y, z) = (1,1,0)$ *and* $(2,3,0)$ *but when* $c = 11$ *and* 13, $(5,1,2)$ *and* $(3,1,1)$ *are also the solutions, respectively together with the common solutions.*

Proof: We prove the theorem in following cases.

Case I: $c = 11$

For $z \geq 3$, we find by using *mod* 1331 that $3^{55} \equiv 1 (\text{mod}\, 1331)$, which implies $x \equiv 0 (\text{mod}\, 55)$. But, if we use *mod* 23, we also get $3^{55} \equiv 1 (\text{mod}\, 23)$, which is a contradiction.

Suppose $y \geq 4$, we find by using *mod* 16 that $3^4 \equiv 1 (\text{mod}\, 16)$ so that $x \equiv 0 (\text{mod}\, 4)$. But, if we use *mod* 5, we also get $3^4 \equiv 1 (\text{mod}\, 5)$, which is a contradiction and so there is no solution.

Therefore,

If $z = 2$ and $y \leq 3$, we find the solutions $(x, y, z) = (5,1,2)$.

Case II: $c = 13$

For $z \geq 2$, using *mod* 169, we obtain $x \equiv 0 \ (\text{mod}\, 39)$. Further, using *mod* 313, we obtain a contradiction.

Suppose $y \geq 4$, using *mod* 16, we find $x \equiv 0 \ (\text{mod}\, 4)$. This yields a contradiction for *mod* 5 and so there is no solution.

Therefore,

If $z = 1$ and $y \leq 3$, we find the solutions $(x, y, z) = (3,1,1)$.

Case III: $c = 17$

For $z \geq 1$, using *mod* 17, we find $x \equiv 0 \ (\text{mod}\, 16)$. Now, if we use *mod* 5, we find a contradiction.

Considering $y \geq 4$, using *mod* 16, we obtain $x \equiv 0 \ (\text{mod}\, 4)$. In this case we have a contradiction for *mod* 5 and so there is no solution.

Case IV: $c = 19$

For $z \geq 1$, using *mod* 19, we find $x \equiv 0 \ (\text{mod}\, 18)$. But, if we use *mod* 7, we obtain a contradiction.

Suppose $y \geq 4$, using *mod* 16, we find $x \equiv 0 \ (\text{mod}\, 4)$. This yields a contradiction for *mod* 5 and so there is no solution.

Case V: $c = 23$

For $z \geq 1$, using *mod* 23, we find $x \equiv 0 \ (\text{mod}\, 11)$. Now, if we use *mod* 3851, we find a contradiction.

Suppose $y \geq 4$, using *mod* 16, we obtain $x \equiv 0 \ (\text{mod}\, 4)$. This yields a contradiction for *mod* 5 and so there is no solution.

Therefore, we conclude from the above cases that,

If $z = 0$ and $y \leq 3$, we find the solutions $(x, y, z) = (1,1,0)$ and $(2,3,0)$,

and there is no other solution for $c \in \{11,17,19,23\}$ when $z = 1$ and $y \leq 3$.

Theorem 4. *For the prime numbers* $c \in \{11,13,17,19,23\}$ *the Diophantine equation*

$$3^x - 2^y c^z = -1$$

has nonnegative integral solutions: $(x, y, z) = (1,2,0)$ *and* $(0,1,0)$.

Proof: We prove the theorem in following cases.

Case 1: $c = 11$

For $z \geq 1$, using *mod* 11, we find

$$3^x \equiv 3,9,5,4,1 \ (\text{mod}\, 11)$$

and $$2^y 11^z \equiv 0 \ (\text{mod}\, 11)$$

but $$-1 \equiv -1 \ (\text{mod}\, 11) \equiv 10 \ (\text{mod}\, 11),$$

which is a contradiction and so there is no solution.

Considering $y \geq 4$, using *mod* 16, we find

$$3^x \equiv 3,9,11,1 \pmod{16}$$

and $$2^y 11^z \equiv 0 \pmod{16}$$

but $$-1 \equiv -1 \pmod{16} \equiv 15 \pmod{16}$$

which yields a contradiction and so there is no solution.

Case II: $c = 13$

For $z \geq 1$, using *mod* 13, we find

$$3^x \equiv 3,9,1 \pmod{13}$$

and $$2^y 13^z \equiv 0 \pmod{13}$$

but $$-1 \equiv -1 \pmod{13} \equiv 12 \pmod{13}$$

This yields a contradiction and so there is no solution for $z \geq 1$.

For $y \geq 4$, using *mod* 16, we find

$$3^x \equiv 3,9,11,1 \pmod{16}$$

and $$2^y 13^z \equiv 0 \pmod{16}$$

but $$-1 \equiv -1 \pmod{16} \equiv 15 \pmod{16},$$

which is a contradiction and so there is no solution.

Case III: $c = 17$

For $z \geq 1$, using *mod* 17, we find $x \equiv 0 \pmod{8}$. But, if we use *mod* 193, we find a contradiction.

For $y \geq 4$, using *mod* 16, we find

$$3^x \equiv 3,9,11,1 \pmod{16}$$

and $$2^y 17^z \equiv 0 \pmod{16}$$

but $$-1 \equiv -1 \pmod{16} \equiv 15 \pmod{16}$$

This yields a contradiction and so there is no solution.

Case IV: $c = 19$

For $z \geq 1$, using *mod* 19 we obtain $x \equiv 0 \pmod{9}$. Now, if we use *mod* 7, we find a contradiction.

For $y \geq 4$, using *mod* 16 we find

$$3^x \equiv 3,9,11,1 \pmod{16}$$

and $$2^y 19^z \equiv 0 \pmod{16}$$

but $$-1 \equiv -1 \pmod{16} \equiv 15 \pmod{16},$$

which is a contradiction and this yields that there is no solution.

Case V: $c = 23$

For $z \geq 1$, using *mod* 23, we find

$$3^x \equiv 3,9,4,12,13,16,2,6,18,8,1 \pmod{23}$$

and $$2^y 23^z \equiv 0 \pmod{23}$$

but $$-1 \equiv -1 \pmod{23} \equiv 22 \pmod{23},$$

which contradicts and yields that there is no solution for $z \geq 1$.

Considering $y \geq 4$, using *mod* 16, we find

$$3^x \equiv 3,9,11,1 \pmod{16}$$

and $$2^y 23^z \equiv 0 \pmod{16}$$

but $$-1 \equiv -1 \pmod{16} \equiv 15 \pmod{16}$$

This yields a contradiction and so there is no solution.

Therefore, we conclude from the above cases that,
If $z=0$ and $y \le 3$, we find the solutions $(x, y, z) = (1,2,0)$ and $(0,1,0)$.
If $z=1$ and $y \le 3$, there is no other solution.

Conclusion

We have solved the Diophantine equation $2^x - 3^y c^z = \pm 1$ for $c \in \{23, 29, 31, 37, 41\}$ and $3^x - 2^y c^z = \pm 1$ for $c \in \{11, 13, 17, 19, 23\}$.
The nonnegative integral common solution of $2^x - 3^y c^z = 1$ for $c \in \{23, 29, 31, 37, 41\}$ are $(1,0,0)$ and $(2,1,0)$ but when $c = 31$, $(5,0,1)$ is also a solution, together with the common solution.
The equation $2^x - 3^y c^z = -1$ for $c \in \{23, 29, 31, 37, 41\}$ has nonnegative integral solutions: $(1,1,0)$ and $(3,2,0)$.
The nonnegative integral common solution of $3^x - 2^y c^z = 1$ for $c \in \{11, 13, 17, 19, 23\}$ are $(1,1,0)$ and $(2,3,0)$ but when $c = 11$ and 13, $(5,1,2)$ and $(3,1,1)$ are also the solutions, respectively together with the common solution.
And the equation $3^x - 2^y c^z = -1$ for $c \in \{11, 13, 17, 19, 23\}$ has nonnegative integral solutions: $(1,2,0)$ and $(0,1,0)$.
The solution of this Diophantine equation for other prime numbers is also possible, but we have left that for future work.

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